nances are obtained by equating $\boldsymbol{\varepsilon}'$ to $\boldsymbol{\mu}$ and the permeability μ_0 of the medium around the sphere to ϵ_0/ϵ_{zz} in the equations for the magnetostatic resonances. By using this analogy and the characteristic equation for magnetostatic resonance,[5] we obtain the characteristic equation for electrostatic plasma resonance:

$$(n+1)\epsilon_0/\epsilon_{zz}+\xi[P_n{}^m(\xi)]'/P_n{}^m(\xi)=\pm m\nu,$$

where

$$\xi^2=1-\frac{\epsilon_{zz}\omega^2(\omega^2-\omega_c{}^2)}{\omega_p{}^2\omega_c{}^2},$$

$$\nu=\frac{\omega_p{}^2}{(\omega^2-\omega_c{}^2)}\frac{\omega_c}{\omega\epsilon_{zz}}.$$

This characteristic equation reduces to Eq. (6) for $n=m$. For $n=2$ and $\epsilon_0=1$ the characteristic equations are

$$\begin{aligned} y(y\pm\alpha)&=1, \quad (|m|=2)\\ y^3\mp\alpha y^2-y\pm\tfrac{1}{2}\alpha&=0, \quad (|m|=1)\\ 2\epsilon_L y^4-y^2[2(1+\alpha^2)\epsilon_L+3+2\epsilon_L]&\\ +\alpha^2(1+2\epsilon_L)+(3+2\epsilon_L)&=0, \quad (m=0) \end{aligned} \tag{19}$$

where $y=(\omega/\omega_p)(\frac{3}{2}+\epsilon_L)^{1/2}$ and $\alpha=(\omega_c/\omega_p)(\frac{3}{2}+\epsilon_L)^{1/2}$.

Figure 4 shows the solutions of Eqs. (19) as a function of α. These solutions for $|m|=2$ and $|m|=1$ are independent of ϵ_L. For $m=0$, ϵ_L appears explicitly in the characteristic equation. These solutions have been plotted for $\epsilon_L\to\infty$. In the experiments reported here $\omega\ll\omega_p$ and only $m=1$ and $m=2$ resonances could appear. The $m=1$ resonance, however, would appear at a very low magnetic field. Because of collision broadening it looks like a smooth decrease in the conductivity rather than like a resonance. The $|m|=2$ resonance is very close to the uniform magnetoplasma mode.

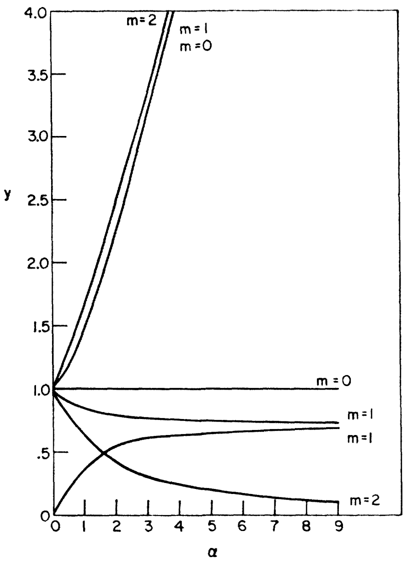

FIG. 4. Reduced frequency y vs reduced magnetic field α [defined by Eq. (19)] for the electrostatic modes of a plasma sphere for $n=2$.

PHYSICAL REVIEW VOLUME 129, NUMBER 3 1 FEBRUARY 1963

Oscillations of a Plasma in a Magnetic Field

M. J. STEPHEN
Physics Department, Yale University, New Haven, Connecticut
(Received 14 September 1962)

The plasma oscillations of a high-temperature and a degenerate plasma, both with and without an applied magnetic field, are discussed using a Green function technique. For the high-temperature plasma in lowest order the results reduce to those obtained via the linearized Boltzmann-Vlasov equation in the classical limit. The quantum-mechanical effects are important for strong fields where the quantization of the orbits of the electrons in the field must be taken into account. The dispersion relation of a degenerate plasma in a magnetic field is obtained within the random phase approximation. This dispersion relation is discussed for various special cases. The fluctuation spectrum of the plasma is obtained by making use of the fluctuation dissipation theorem.

1. INTRODUCTION

IN this paper the small oscillations of an electron gas in thermodynamic equilibrium are discussed. The two cases of a plasma in zero external magnetic field and a plasma situated in a strong uniform magnetic field are considered. The presence of the ions is neglected and we use the simple model where the ions are smeared out into a compensating positive background. The usual treatments of the oscillations of a high-temperature plasma are based on the collisionless Boltzmann-Vlasov (B.V.) equation or on the hydrodynamic equations of motion. Reviews of recent work in this field have been written by Thompson[1] and Oster.[2] Here we use a different ap-

[1] W. B. Thompson, *Reports on Progress in Physics* (The Physical Society, London, 1961), Vol. 24, p. 363.
[2] L. Oster, Rev. Mod. Phys. 32, 141 (1960).

proach based on the use of Green functions. This work follows on from some earlier work by Montroll and Ward[3] on the electron gas and by the present author on the magnetic properties of an electron gas.[4]

In lowest order in the electron interaction, for zero magnetic field, the plasma dispersion relation obtained by this method is identical with that from the quantum mechanical form of the linearized B.V. equation[5] and reduces to the ordinary linearized B.V. equation result in the classical limit $\hbar \rightarrow 0$ (or the long-wavelength limit). When the plasma is situated in a uniform magnetic field, the plasma dispersion relation has been discussed by Gross[6] and Bernstein[7] by means of the B.V. equation. The Green function method leads naturally to the quantum-mechanical generalization of their results. These quantum-mechanical effects are only important for strong magnetic fields when the quantization of the orbits of the electrons in the field must be taken into account. The relevant parameter is $\hbar\omega_c\beta$, where $\omega_c = eH/mc$ and is the cyclotron frequency of an electron in the magnetic field. Thus, for the interesting case of a high-temperature plasma, these effects are negligible but as they enter naturally into the results they are retained.

The spectrum of density fluctuations for real frequencies of a classical plasma, both with and without a magnetic field, has been discussed by Salpeter.[8] Making use of the fluctuation dissipation theorem the quantum-mechanical generalizations of Salpeter's results are obtained.

Plasma oscillations in a degenerate Fermi gas have been considered by Bohm and Pines and many other authors. The random phase approximation (RPA) (or linearized equations of motion) gives results analogous to those obtained from the linearized B.V. equation. Corrections to these results have been obtained by Hubbard, DuBois, Kanazawa *et al.*, and others.[9] Here we consider a degenerate plasma in a uniform magnetic field in the random phase approximation. The dispersion relation of a degenerate plasma in a uniform field has recently been considered by Zyryanov.[10] He makes some unnecessary approximations in the calculation of matrix elements and here we give an exact treatment (within the RPA). These results have application in the theory of magnetoacoustic resonance which is considered elsewhere.

In Sec. 2 we discuss the generalized dielectric constant of the system. From a knowledge of this quantity we can obtain the plasma dispersion relation and also the spectrum of fluctuations of the plasma. In Sec. 3 we obtain results for a high-temperature, low-density plasma and in Sec. 4 for a degenerate, high-density plasma.

2. GENERALIZED DIELECTRIC CONSTANT

The work of Lindhard, Hubbard, and Nozières and Pines[11] has demonstrated the importance of the concept of a generalized dielectric constant $\boldsymbol{\varepsilon}(\mathbf{q},\omega)$ for a wave vector $\mathbf{q}$ and a frequency ω for a system of interacting particles and its relation to the plasma oscillations. The quantity $\boldsymbol{\varepsilon}(\mathbf{q},\omega)$ is also closely related to the Fourier transform of the density correlation function of van Hove[12] and thus to the fluctuation spectrum in the gas.

Consider the Fourier transform of Poisson's equation:

$$\mathbf{q}\cdot\boldsymbol{\varepsilon}(\mathbf{q},\omega)\cdot\mathbf{q}\Phi(\mathbf{q},\omega)=4\pi\rho(\mathbf{q},\omega). \tag{2.1}$$

This equation has the important feature that if $\mathbf{q}\cdot\boldsymbol{\varepsilon}(\mathbf{q},\omega)\cdot\mathbf{q}$ vanishes for some frequency, $\omega(\mathbf{q})$ say, then $\Phi(\mathbf{q},\omega)$ can have any value. This means that the system can sustain oscillations at the angular frequency $\omega(\mathbf{q})$. In general, $\mathbf{q}\cdot\boldsymbol{\varepsilon}(\mathbf{q},\omega)\cdot\mathbf{q}$ is complex and will only vanish at a point in the complex ω plane giving rise to damped oscillations.

In what follows we neglect magnetic interactions between the particles and so we are only concerned with the longitudinal dielectric constant. In the case where there is an external magnetic field we neglect coupling between transverse and longitudinal oscillations. This requires that $qc \gg \omega_p$ and ω_c where $\omega_p = 4\pi\rho e^2/m$ and $\omega_c = eH/mc$.

To illustrate the concept of $\boldsymbol{\varepsilon}(\mathbf{q},\omega)$, consider the scattering of two electrons in the gas. The simplest idea is that these electrons interact via the bare Coulomb interaction. But in considering this process alone we are neglecting the effects of the surrounding medium. An electron can also interact with the medium, and the resulting polarization charge (or induced charge) can then act on the other electron via the Coulomb interaction. In terms of diagrams this means that besides the bare Coulomb line we should also consider all those proper polarization diagrams (PPD). These are defined to be all those diagrams with two external Coulomb lines and which cannot be broken up into simpler diagrams by cutting a single interaction line. We denote the sum of all these diagrams by Q. Some simple examples are shown in Fig. 1.

It is simpler for the present applications to work with imaginary times, i.e., temperatures and the effective interaction between particles $V(\mathbf{r}_1-\mathbf{r}_2, \beta_1-\beta_2)$ is then

---

[3] E. W. Montroll and J. Ward, Phys. Fluids **1**, 51 (1958).
[4] M. J. Stephen, Proc. Roy. Soc. (London) **A 265**, 215 (1962) (referred to as I).
[5] O. von Roos, Phys. Rev. **119**, 1174 (1960).
[6] E. P. Gross, Phys. Rev. **82**, 232 (1951).
[7] I. Bernstein, Phys. Rev. **109**, 10 (1958).
[8] E. E. Salpeter, Phys. Rev. **120**, 1528 (1960); **122**, 1663 (1961).
[9] J. Hubbard, Proc. Roy. Soc. (London) **A 240**, 539 (1957); D. F. DuBois, Ann. Phys. (N. Y.) **7**, 174 (1959); H. Kanazawa, S. Misawa, and E. Fujita, Progr. Theoret. Phys. (Kyoto) **23**, 426 (1960).
[10] P. S. Zyryanov, Soviet Phys.—JETP **13**, 751 (1961).
[11] J. Lindhard, Kgl. Danske Videnskab. Selskab, Mat-Fys. Medd. **28**, No. 8 (1954); J. Hubbard, Proc. Phys. Soc. (London) **A 68**, 976 (1955); P. Nozières and D. Pines, Nuovo Cimento **9**, 470 (1958).
[12] L. van Hove, Phys. Rev. **95**, 249 (1954).

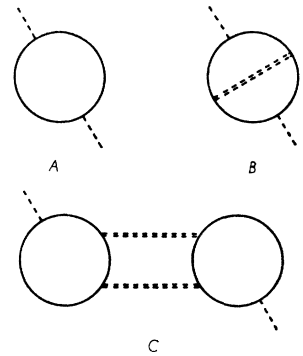

FIG. 1. Proper polarization diagrams. Dashed line: Coulomb interaction; double dashed line: effective interaction.

the solution of the integral equation (see Hubbard[9])

$$V(1-2)=u(1-2)-\int u(1-3)Q(3-4)V(4-2)d3d4, \quad (2.2)$$

where we have used the abbreviation $1=(\mathbf{r}_1,\beta_1)$ and $u(1-2)$ is the bare Coulomb interaction

$$u(1-2)=u(\mathbf{r}_1-\mathbf{r}_2)\delta(\beta_1-\beta_2)=(e^2/r_{12})\delta(\beta_1-\beta_2). \quad (2.3)$$

We require all quantities to have period β in temperature space and will thus be only considered in the interval $0-\beta$. Thus, Fourier transforms are introduced in the form

$$V(1-2)=\frac{1}{(2\pi)^3}\frac{1}{\beta}\sum_{n=-\infty}^{\infty} e^{2\pi i n\beta_{12}/\beta}\times\int d\mathbf{q}\, e^{i\mathbf{q}\cdot\mathbf{r}_{12}}V(\mathbf{q},n), \quad (2.4)$$

where n is an integer. Then from (2.2)

$$V(\mathbf{q},n)=\frac{u(\mathbf{q})}{1+u(\mathbf{q})Q(\mathbf{q},n)}. \quad (2.5)$$

Comparison of (2.5) with Poisson's equation (2.1) gives

$$\frac{\mathbf{q}\cdot\boldsymbol{\varepsilon}(\mathbf{q},n)\cdot\mathbf{q}}{q^2}=1+\frac{4\pi e^2}{q^2}Q(\mathbf{q},n). \quad (2.6)$$

To obtain real frequencies ω, we must eventually make the analytic continuation $2\pi i n\to\hbar\omega\beta$. This will be discussed below. The plasma dispersion relation is thus

$$1+(4\pi e^2/q^2)Q(\mathbf{q},n)=0. \quad (2.7)$$

We are also interested in the spectrum of density fluctuations, i.e., the intensity $F(\mathbf{q},\omega)$ of the density fluctuation of the gas in thermodynamic equilibrium with wave vector $\mathbf{q}$ and frequency (real) ω. $F(\mathbf{q},\omega)$ is the Fourier transform of the density correlation function

$$F(\mathbf{r}_1-\mathbf{r}_2,\, t_1-t_2)=\langle\{\rho(\mathbf{r}_1,t_1),\rho(\mathbf{r}_2,t_2)\}_+\rangle, \quad (2.8)$$

where $\rho(\mathbf{r},t)$ is the density operator in the Heisenberg representation and the angular brackets indicate a trace.

$$\langle\mathrm{Op}\rangle=\mathrm{Tr}[D\,\mathrm{Op}]/\mathrm{Tr}[D], \quad (2.9)$$

where D is the density matrix. $F(\mathbf{q},\omega)$ is related to the linear response function R of the system via the fluctuation dissipation theorem. The response function is defined by

$$R(\mathbf{r}_1-\mathbf{r}_2,\, t_1-t_2)=\langle[\rho(\mathbf{r}_1,t_1),\rho(\mathbf{r}_2,t_2)]_-\rangle. \quad (2.10)$$

By writing out the trace explicitly in (2.8) and (2.10) it is easy to show that (Kubo[13])

$$F(\mathbf{q},\omega)=\coth(\hbar\omega\beta/2)R(\mathbf{q},\omega). \quad (2.11)$$

In turn the response function is related to the generalized dielectric constant of the system. A very good discussion has been given by Nozières and Pines[11] and we only quote the result, which is required later, relating the spectrum of density fluctuations to the dielectric constant. We generalize their result to a finite temperature.

$$F(\mathbf{q},\omega)=-\frac{\hbar q^2}{2\pi}\coth\left(\frac{\hbar\omega\beta}{2}\right)\mathrm{Im}\frac{q^2}{\mathbf{q}\cdot\boldsymbol{\varepsilon}_r(\mathbf{q},\omega)\cdot\mathbf{q}}. \quad (2.12)$$

Note that in (2.12) the retarded dielectric constant, $\boldsymbol{\varepsilon}_r(\mathbf{q},\omega)$ occurs.

We now apply these results to an electron plasma in equilibrium. The central problem is the calculation of the dielectric constant. We consider separately the cases of a high-temperature, low-density plasma and a degenerate plasma.

3. HIGH-TEMPERATURE, LOW-DENSITY PLASMA

We assume that the electrons obey Boltzmann statistics. The lowest order PPD is shown in Fig. 1(a) and the contribution of this diagram to $Q(\mathbf{q},n)$ is $Q_1(\mathbf{q},n)$ say, and is given by (in the notation of I)

$$Q_1(\mathbf{q},n)=z\Lambda_1(\mathbf{q},n), \quad (3.1)$$

where z is the fugacity and

$$\Lambda_1(\mathbf{q},n)=\int_0^\beta d\beta_{21}\, e^{-2\pi i n\beta_{21}/\beta}\Lambda_1(\mathbf{q},\beta_{21}), \quad (3.2)$$

$$\Lambda_1(\mathbf{q},\beta_{21})=\frac{1}{(2\pi)^3}\left(\frac{\pi}{a^2}\right)^{3/2}\frac{2\cosh b}{b^{1/2}\sinh b}\times\exp\left\{-a^2\left[q_{\parallel}^2 b_{21}\left(1-\frac{b_{21}}{b}\right)+q_{\perp}^2\frac{\sinh b_{21}\sinh(b-b_{21})}{\sinh b}\right]\right\}. \quad (3.3)$$

[13] R. Kubo, J. Phys. Soc. Japan **12**, 570 (1957).

In (3.3)

$$a^2=\hbar/m\omega_c,\quad b=\tfrac{1}{2}\hbar\omega_c\beta,\quad b_{21}=\tfrac{1}{2}\hbar\omega_c\beta_{21},$$

and $q_{\parallel}$ and $q_{\perp}$ are the components of $\mathbf{q}$ parallel and perpendicular to the uniform field H, respectively. The factor 2 cosh$b$ arises from spin. We require the general result (3.3) later, but we begin by considering the case of zero magnetic field.

A. Zero Magnetic Field

Taking the limit as $H\rightarrow 0$ in (3.3) and substituting in (3.2) we obtain the plasma dispersion relation from (2.7) (introducing $x=\beta_{21}/\beta$)

$$1+\frac{4\pi e^2 z_0\beta}{q^2}\int_0^1 dx\, e^{-2\pi inx}\exp[-\lambda^2q^2x(1-x)]=0,\quad (3.4)$$

where

$$\lambda^2=\frac{\hbar^2\beta}{2m}\quad\text{and}\quad z_0=\frac{2}{(2\pi)^3}\left(\frac{\pi}{\lambda^2}\right)^{3/2}.\quad (3.5)$$

To obtain the dispersion relation in the long-wavelength limit, i.e., $q\rightarrow 0$, we can simply expand the integrand of (3.4) in powers of q^2 and after integration make the substitution $2\pi in\rightarrow\hbar\omega\beta$. On solving for the plasma frequency ω (replacing z_0 by ρ, the particle density which is permissible in this approximation to Q), we obtain the well-known result

$$\omega^2=\omega_p^2+3q^2/m\beta,\quad \omega_p^2=4\pi e^2\rho/m.\quad (3.6)$$

To obtain more general results we rewrite the integral in (3.4) as

$$\left(\frac{\lambda^2}{\pi}\right)^{1/2}\int_0^1 dx\int_{-\infty}^{\infty}dk\, e^{-2\pi inx}\times\exp\{-\lambda^2[k^2x+(k+q)^2(1-x)]\}.\quad (3.7)$$

Performing the integration over x, the dispersion relation (3.4) becomes

$$1=\frac{\omega_p^2}{q^2}\left(\frac{\lambda^2}{\pi}\right)^{1/2}\int_{-\infty}^{\infty}dk\frac{(2kq+q^2)}{\omega^2-(\hbar/2m)^2(2kq+q^2)^2}e^{-\lambda^2k^2}.\quad (3.8)$$

This is exactly the dispersion relation obtained from the quantum mechanical form of the linearized B.V. equation. In (3.8) ω is not necessarily real and, in fact, there is no solution for real ω as in that case the integral on the right-hand side would diverge. By working with imaginary times (i.e., temperature) we avoid the difficulties which arise for real frequencies. Writing $\omega=\omega_r+i\sigma$ where ω_r and σ are both real, it is easy to determine them in the limit $\sigma\rightarrow 0$. We find that ω_r is given by (3.6) in the long-wavelength limit and

$$\sigma=-\pi^{1/2}(\tfrac{1}{2}m\beta)^{3/2}\frac{\omega_p^4}{q^3}\exp\left(-\frac{m\omega_p^2\beta}{2q^2}\right).\quad (3.9)$$

This is the familiar Landau damping of plasma oscillations. Equation (3.8) has been widely discussed in the literature and we will not discuss it further. Note that in the limit $\hbar\rightarrow 0$ (or long-wavelength limit), it reduces exactly to the result of the linearized B.V. equation. To obtain this limit the new variable $u=\hbar k/m+\hbar q/2m$ is introduced in (3.8) and then $\hbar$ is set equal to 0.

B. Nonzero Magnetic Field

In this case from (3.3) the dispersion relation is

$$1=-\frac{m\beta\omega_p^2}{q^2}\int_0^1 dx\, e^{-2\pi inx}\exp\left[-\lambda^2q_{\parallel}^2x(1-x)-a^2q_{\perp}^2\frac{\sinh bx\,\sinh b(1-x)}{\sinh b}\right].\quad (3.10)$$

For long wavelengths we can simply expand the exponential as before and the plasma frequency is found to be the solution of

$$1=\frac{q_{\parallel}^2\omega_p^2}{q^2\omega^2}+\frac{q_{\perp}^2\omega_p^2}{q^2(\omega^2-\omega_c^2)}+O(q^2),\quad (3.11)$$

a well-known result. When $q_{\perp}=0$, i.e., propagation parallel to the field, (3.10) just reduces to (3.4). The motion of the particles along the lines of force is unaffected by the field. For propagation perpendicular to the field $q_{\parallel}=0$ and we find on retaining terms of order q^2 in (3.10) that

$$\omega_{\perp}^2=\omega_p^2+\omega_c^2+\frac{\hbar q_{\perp}^2\omega_p^2}{2m\omega_c}\coth(\tfrac{1}{2}\hbar\omega_c\beta)\frac{3\omega_c^2}{\omega_p^2-3\omega_c^2}.\quad (3.12)$$

This reduces to the result obtained via the B.V. equation when $\hbar\omega_c\beta\ll 1$. It is interesting to note the effect of the exact quantum-mechanical treatment of the magnetic field leading to the third term in (3.12).

In order to obtain the complete dispersion relation we use (3.7) to rewrite the first part of the exponential in (3.10). For the second part we use

$$\sinh bx\,\sinh b(1-x)=\tfrac{1}{2}\cosh b-\tfrac{1}{2}\cosh b(1-2x)$$

and then substitute

$$\exp\left[\frac{a^2q_{\perp}^2}{2\sinh b}\cosh b(1-2x)\right]=\sum_{n=-\infty}^{\infty}I_n\left[\frac{a^2q_{\perp}^2}{2\sinh b}\right]e^{nb(1-2x)},\quad (3.13)$$

where I_n is a modified Bessel function of the first kind.

Equation (3.13) follows from the result[14]

$$e^{ix\sin\theta}=\sum_{n=-\infty}^{\infty} J_n(x)e^{in\theta}.$$

The integration over x in (3.10) can now be carried out and after some rearrangement we obtain the dispersion relation

$$1=\frac{\hbar\omega_p^2}{2\pi q^2}\left(\frac{\pi}{\lambda^2}\right)^{1/2} e^{-\frac{1}{2}a^2q_\perp^2\coth b}\int_{-\infty}^{\infty} dk\, e^{-\lambda^2k^2} \times\sum_{n=-\infty}^{\infty} I_n\left[\frac{a^2q_\perp^2}{2\sinh b}\right]\left[\frac{e^{nb}}{\Omega_n^-}-\frac{e^{-nb}}{\Omega_n^+}\right], \quad (3.14)$$

where

$$\Omega_n^\pm=\omega+(\hbar/2m)(2kq_{\parallel}\pm q_{\parallel}^2)-n\omega_c.$$

In the classical limit this reduces to the result of Gross[6] and Bernstein.[7] When the direction of propagation of the plasma wave is perpendicular to the field the integral in (3.14) can be done in an elementary manner and we get

$$1=\frac{4\omega_p^2}{a^2q^2}e^{-\frac{1}{2}a^2q_\perp^2\coth b}\sum_{n=1}^{\infty} I_n\left[\frac{a^2q_\perp^2}{2\sinh b}\right]\frac{n\sinh nb}{\omega^2-n^2\omega_c^2}. \quad (3.15)$$

This type of dispersion relation has been studied in detail by Bernstein and Salpeter.[8] Their results are valid for $\hbar\omega_c\beta\ll 1$ and (3.14) and (3.15) are the generalizations to arbitrary values of this parameter. It is also felt that the present derivation is simpler and free from the objections raised against Bernstein's derivation.[2]

In the case of (3.14) the solution ω cannot be real and we get again the phenomenon of Landau damping. This damping vanishes as $q_{\parallel}\rightarrow 0$ and it is possible to show that the solutions of (3.15) are real. The presence of the magnetic field inhibits the random thermal motions of the particles which normally give rise to the Landau damping. As $\omega/\omega_c\rightarrow\infty$ the right-hand side of (3.15) is essentially zero for $n<\omega/\omega_c<n+1$ and lies very close to the line $\omega=n\omega_c$ and so the solutions are approximately $\omega\sim n\omega_c$, i.e., multiples of the cyclotron frequency of the electron. The right-hand side of (3.15) varies between $+\infty$ and $-\infty$ in the range $n<\omega/\omega_c<n+1$ and the negative values correspond to gaps in the frequency spectrum, i.e., waves which will not propagate. These frequency gaps have been discussed by Bernstein and are considered below in the case of the degenerate gas.

We can obtain an alternate form for the dispersion relation which is useful for strong fields by using the result (Watson[15])

$$e^{-\frac{1}{2}a^2q^2\coth b}I_n\left[\frac{a^2q^2}{2\sinh b}\right] =2\sinh b(\tfrac{1}{2}a^2q^2)^n e^{-\frac{1}{2}a^2q^2}\sum_{s=0}^{\infty}\frac{s!}{(n+s)!} \times[L_s^n(\tfrac{1}{2}a^2q^2)]^2e^{-(n+2s+1)b}, \quad (3.16)$$

where L_s^n is a Laguerre polynomial. On substituting in (3.15) we obtain an alternate form of the dispersion relation useful for $a^2q^2\ll 1$.

$$1=\frac{8\omega_p^2\sinh b}{a^2q^2}e^{-\frac{1}{2}a^2q_\perp^2}\sum_{n=1}^{\infty}\sum_{s=0}^{\infty}\frac{s!}{(n+s)!}(\tfrac{1}{2}a^2q_\perp^2)^n \times[L_s^n(\tfrac{1}{2}a^2q_\perp^2)]^2\frac{n\sinh nb}{\omega^2-n^2\omega_c^2}e^{-(n+2s+1)b}. \quad (3.17)$$

C. Spectrum of Fluctuations

The fluctuation spectrum is expressed by (2.12) in terms of the imaginary part of the retarded, inverse dielectric constant. In order to obtain this from our previous expressions we must replace ω by $\omega+i\eta$ where ω is a real frequency and η a small positive constant. This replacement ensures that the poles of $\boldsymbol{\varepsilon}_r(\mathbf{q},\omega)$ all lie in the lower half-plane corresponding to the definition of the retarded response of the system.

In the absence of the magnetic field $\boldsymbol{\varepsilon}_r(\mathbf{q},\omega)$ can be obtained from (3.8) and on substitution in (2.12) we find

$$F(\mathbf{q},\omega)=\frac{m\omega_p^2}{q}\left(\frac{m\beta}{2\pi}\right)^{1/2}\cosh(\tfrac{1}{2}\hbar\omega\beta)\,|1-G(\mathbf{q},\omega)|^{-2} \times\exp\left[-\frac{m\omega^2\beta}{2q^2}-\tfrac{1}{4}\lambda^2q^2\right], \quad (3.18)$$

where $G(\mathbf{q},\omega)$ is the right-hand side of (3.8) with ω replaced by $\omega+i\eta$. In the long-wavelength limit $F(\mathbf{q},\omega)$ reduces correctly to

$$F(\mathbf{q},\omega)\underset{q\rightarrow 0}{\longrightarrow}(q^2/2\beta)\delta(\omega-\omega_p),$$

a sharp line at the plasma frequency. The result of Salpeter is obtained from (3.18) when $\hbar\omega\beta\ll 1$ and $\lambda^2q^2\ll 1$. This leads to some modification of the fluctuation spectrum at high frequencies and short wavelengths.

In the presence of a magnetic field we use (3.14) to

[14] G. N. Watson, *Treatise on the Theory of Bessel Functions* (Cambridge University Press, New York, 1944).

[15] G. N. Watson, J. London Math. Soc. **8**, 189 (1938).

obtain $\boldsymbol{\varepsilon}_r$ and find

$$F(\mathbf{q},\omega)=\frac{m\omega_p^2}{q_{\shortparallel}}\left(\frac{m\beta}{2\pi}\right)^{1/2}\cosh(\tfrac{1}{2}\hbar\omega\beta)$$
$$\times e^{-\frac{1}{2}a^2q_\perp^2\coth b}|1-H(\mathbf{q},\omega)|^{-2}\sum_{n=-\infty}^{\infty}I_n\left[\frac{a^2q_\perp^2}{2\sinh b}\right]$$
$$\times\exp\left[-\frac{m\beta}{2q_{\shortparallel}^2}(\omega-n\omega_c)^2-\tfrac{1}{4}\lambda^2q_{\shortparallel}^2\right],\quad(3.19)$$

where $H(\mathbf{q},\omega)$ is the right-hand side of (3,14) with ω replaced by $\omega+i\eta$. As $q_{\shortparallel}\rightarrow 0$, corresponding to propagation perpendicular to the field, we get a series of sharp lines in the spectrum at $\omega=n\omega_c$. In the limit $\hbar\omega\beta\ll 1$, $\hbar\omega_c\beta\ll 1$, and $\lambda^2q_{\shortparallel}^2\ll 1$ (3.19) reduces to the result of Salpeter which was obtained from the linearized Boltzmann-Vlasov equation.

The damping of the plasma oscillations can be obtained immediately from these results. If we write $\omega=\omega_r+i\sigma$ where ω_r and σ are both real, then for small σ and long wavelengths (for simplicity we omit the subscript on ω_r)

$$\sigma(\omega)=-\frac{\omega^3}{2\omega_p^2}\operatorname{Im}\frac{\mathbf{q}\cdot\boldsymbol{\varepsilon}(\mathbf{q},\omega)\cdot\mathbf{q}}{q^2}.\quad(3.20)$$

In obtaining this result we have supposed that

$$\operatorname{Re}\mathbf{q}\cdot\boldsymbol{\varepsilon}(\mathbf{q},\omega)\cdot\mathbf{q}/q^2=1-\omega_p^2/\omega^2.$$

Then from (3.19) and (3.20) we get

$$\sigma(\omega)=-\frac{\pi m\omega^3}{\hbar q^2q_{\shortparallel}}\left(\frac{m\beta}{2\pi}\right)^{1/2}\sinh(\tfrac{1}{2}\hbar\omega\beta)$$
$$\times e^{-\frac{1}{2}a^2q_\perp^2\coth b}\sum_{n=-\infty}^{\infty}I_n\left[\frac{a^2q_\perp^2}{2\sinh b}\right]$$
$$\times\exp\left[-\frac{m\beta}{2q_{\shortparallel}^2}(\omega-n\omega_c)^2-\tfrac{1}{4}\lambda^2q_{\shortparallel}^2\right].\quad(3.21)$$

4. DEGENERATE FERMI GAS

We must now include the correct statistics in the calculation of Q. Many authors have considered the dispersion relation in the absence of a magnetic field and we will, therefore, immediately consider the degenerate plasma in a uniform field. We confine the discussion to the RPA, i.e., we only consider the diagram in Fig. 1(a). In the case of Fermi statistics (3.1) is replaced by

$$Q_1(\mathbf{q},n)=\sum_{N=1}^{\infty}(-)^{N+1}z^N\Lambda_N(\mathbf{q},Nn),\quad(4.1)$$

where

$$\Lambda_N(\mathbf{q},Nn)=\int_0^{N\beta}d\beta_{21}e^{-2\pi in\beta_{21}/\beta}\Lambda_N(\mathbf{q},\beta_{21}),\quad(4.2)$$

and

$$\Lambda_N(\mathbf{q},\beta_{21})=\frac{1}{(2\pi)^3}\left(\frac{\pi}{a^2}\right)^{3/2}\frac{2\cosh Nb}{(Nb)^{1/2}\sinh Nb}$$
$$\times\exp\left\{-a^2\left[q_{\shortparallel}^2b_{21}\left(1-\frac{b_{21}}{Nb}\right)+q_\perp^2\frac{\sinh b_{21}\sinh(Nb-b_{21})}{\sinh Nb}\right]\right\}.\quad(4.3)$$

We begin by considering some special cases. For long wavelengths and low fields (4.2) can be evaluated by expansion in powers of q^2 and the magnetic field. The summation over N is done by introducing the Fermi-Dirac integral

$$F_n(\mu)=\frac{1}{\Gamma(n+1)}\int_0^\infty\frac{z^ndz}{1+e^{z-\beta\mu}},$$
$$=\sum_{N=1}^{\infty}(-)^{N+1}N^{-n-1}e^{N\beta\mu},\quad \mu<0,$$
$$=\frac{(\beta\mu)^{n+1}}{\Gamma(n+2)}+\cdots,\quad \mu>0.\quad(4.4)$$

For propagation parallel to the field the plasma frequency is no longer independent of the field because the electron distribution function depends on the field and for long wavelengths and low field we find a small correction

$$\omega_{\shortparallel}^2=\omega_p^2+\frac{6}{5m}q^2\mu_0\left\{1+\frac{5}{16}\left[\left(\frac{\hbar\omega_c}{\mu_0}\right)^2-\frac{1}{3}\left(\frac{\hbar\omega_c}{\mu_0}\right)^2\right]\right\}.\quad(4.5)$$

The first term in the square bracket is due to spin and the second is a small diamagnetic correction arising from the orbital motion of the electrons. In (4.5) μ_0 is the Fermi energy which is related to the density by

$$\mu_0=(3\rho/8\pi)^{2/3}2\pi^2\hbar^2/m.$$

The corresponding result for propagation perpendicular to the field is

$$\omega_\perp^2=\omega_p^2+\omega_c^2+\frac{6}{5m}q^2\mu_0\frac{\omega_p^2}{\omega_p^2-3\omega_c^2}$$
$$\times\left\{1+\frac{5}{16}\left[\left(\frac{\hbar\omega_c}{\mu_0}\right)^2+\frac{2}{3}\left(\frac{\hbar\omega_c}{\mu_0}\right)^2\right]\right\},\quad(4.6)$$

where again the terms in square brackets have been separated into a part due to spin (the first term) and orbital motion.

The limiting case of very strong fields and long wavelengths is also easily discussed. By a strong field we refer to the case where all electrons are in the lowest Landau state and have their spins aligned antiparallel to the field. The chemical potential is now determined in terms of the field and density by

$$\mu_0=\frac{(2\pi)^3\rho^2\pi}{4(\hbar\omega_c)^2}\left(\frac{\hbar^2}{m}\right)^3. \tag{4.7}$$

The plasma frequency is determined for propagation parallel and perpendicular to the field as

$$\begin{aligned}\omega_{\parallel}^2&=\omega_p^2+\frac{2q_{\parallel}^2}{m}\mu_0,\\ \omega_{\perp}^2&=\omega_p^2+\omega_c^2+\frac{3\hbar\omega_c}{2m}q_{\perp}^2\frac{\omega_p^2}{\omega_p^2-3\omega_c^2}.\end{aligned} \tag{4.8}$$

The above results also follow simply from the more general results we will now derive.

Following exactly the steps leading to (3.14) we obtain from (4.2) and (4.3) the dispersion relation

$$\begin{aligned}1=\frac{e^2}{\pi\hbar a^2q^2}\sum_{N=1}^{\infty}(-)^{N+1}e^{\beta\mu N}\coth Nb\,e^{-\frac{1}{2}a^2q_\perp^2\coth Nb}\\ \times\sum_{n=-\infty}^{\infty}I_n\left[\frac{a^2q_\perp^2}{2\sinh Nb}\right]\\ \times\int_{-\infty}^{\infty}dk\,e^{-\lambda^2k^2N}\left[\frac{e^{nNb}}{\Omega_n^-}-\frac{e^{-nNb}}{\Omega_n^+}\right],\end{aligned} \tag{4.9}$$

where $\Omega_n^{\pm}$ is defined below Eq. (3.14) and we have made the usual substitution $2\pi in=\hbar\omega\beta$. If we use (3.16) with b replaced by Nb to simplify (4.9), the sum over N can be done in an elementary manner and the dispersion relation is found to be

$$\begin{aligned}1=\frac{e^2}{\pi\hbar a^2q^2}e^{-\frac{1}{2}a^2q_\perp^2}\sum_{n=0}^{\infty}{}'\sum_{s=0}^{\infty}\int_{-\infty}^{\infty}dk\frac{s!}{(n+s)!}(\tfrac{1}{2}a^2q_\perp^2)^n\\ \times[L_s^n(\tfrac{1}{2}a^2q_\perp^2)]^2\left[\bar f_0(\epsilon_k+\hbar\omega_c(s+\tfrac{1}{2}))\left(\frac{1}{\Omega_n^-}-\frac{1}{\Omega_{-n}^+}\right)\right.\\ \left.+\bar f_0(\epsilon_k+\hbar\omega_c(s+n+\tfrac{1}{2}))\left(\frac{1}{\Omega_{-n}^-}-\frac{1}{\Omega_n^+}\right)\right].\end{aligned} \tag{4.10}$$

For $n=0$ both terms in the square bracket in (4.10) are identical and the prime on the summation indicates that only one term should be retained. In (4.10)

$$\bar f_0(\epsilon_k+\hbar\omega_c(s+\tfrac{1}{2}))=f_0(\epsilon_k+\hbar\omega_c s)+f_0(\epsilon_k+\hbar\omega_c(s+1)),$$

where $f_0(\epsilon_k)$ is the Fermi function $\{1+\exp[\beta(\epsilon_k-\mu)]\}^{-1}$ and $\epsilon_k=\hbar^2k^2/2m$. The function $\bar f_0$ occurs because of the two possible spin orientations in the magnetic field. If spin effects are neglected the $\bar f_0$ should be replaced by f_0.

For propagation parallel to the field (4.10) simplifies to

$$1=\frac{e^2}{\pi ma^2q^2}\sum_{s=0}^{\infty}\int_{-\infty}^{\infty}dk\,\bar f_0(\epsilon_k+\hbar\omega_c(s+\tfrac{1}{2}))\times\frac{2kq+q^2}{\omega^2-(\hbar/2m)^2(2kq+q^2)^2}. \tag{4.11}$$

The magnetic field only enters into the distribution function for the electrons as would be expected from physical considerations. For low fields on replacing the summation in (4.11) by an integration we obtain the well-known Bohm-Pines dispersion relation

$$1=\frac{e^2}{\pi^2mq^2}\int d\mathbf{k}\,f_0(\epsilon_k)\frac{2\mathbf{k}\cdot\mathbf{q}+q^2}{\omega^2-(\hbar/2m)^2(2\mathbf{k}\cdot\mathbf{q}+q^2)^2}. \tag{4.12}$$

For propagation perpendicular to the field (4.10) reduces to

$$\begin{aligned}1=\frac{2e^2}{\pi\hbar a^2q^2}\left(\frac{\pi}{\lambda^2}\right)^{1/2}e^{-\frac{1}{2}a^2q_\perp^2}\sum_{s=0}^{\infty}\sum_{n=1}^{\infty}\frac{s!}{(n+s)!}(\tfrac{1}{2}a^2q_\perp^2)^n\\ \times[L_s^n(\tfrac{1}{2}a^2q_\perp^2)]^2\frac{n\omega_c}{\omega^2-n^2\omega_c^2}[\bar F_{-\frac{1}{2}}(\mu-\hbar\omega_c(s+\tfrac{1}{2}))\\ -\bar F_{-\frac{1}{2}}(\mu-\hbar\omega_c(s+n+\tfrac{1}{2}))].\end{aligned} \tag{4.13}$$

Equation (4.13) is similar in form to the corresponding result for Boltzmann statistics (3.15). It may be shown that the solutions of (4.13) are real corresponding to undamped waves. For large ω/ω_c the solutions lie very close to $\omega\sim n\omega_c$. There are also gaps in the spectrum corresponding to the right-hand side becoming less than zero. We calculate the width of the gap for long wavelengths. Retaining only terms of order q^2 to determine the gap in the range $\omega_c<\omega<2\omega_c$ the form

$$1=\frac{\omega_p^2}{\omega^2-\omega_c^2}+\frac{2q^2\omega_p^2\mu}{5m\omega_c^2}\left[\frac{1}{\omega^2-4\omega_c^2}-\frac{1}{\omega^2-\omega_c^2}\right] \tag{4.14}$$

is sufficient. From (4.14) the width of the gap in this range is found to be

$$\Delta\omega=3q^2\mu/20m\omega_c. \tag{4.15}$$

In the zero-temperature limit the integration in (4.10) can be carried out. We do not write down the answer but consider some special cases of interest that are easily obtained.

(*i*) *Long wavelengths.* In this case we only retain the term $n=0$ in (4.10) and after integration over k and expansion in powers of $q_{\parallel}$ we obtain

$$1=\frac{2e^2q_{\parallel}^2}{\pi a^2q^2\hbar}\sum_{s=0}\left[\frac{V_s}{\omega^2-q_{\parallel}^2V_s^2}+\frac{V_{s+1}}{\omega^2-q_{\parallel}^2V_{s+1}^2}\right], \tag{4.16}$$

where the summation over s is such that $\mu > \hbar\omega_c s$ and

$$V_s^2 = (2/m)(\mu - \hbar\omega_c s).$$

For frequencies such that $\omega^2 > q_{\shortparallel}^2 V_0^2$ (4.16) reduces to (4.5). There are also a series of solutions for ω which are real in the range $q_{\shortparallel}^2 V_s^2 > \omega^2 > q_{\shortparallel}^2 V_{s+1}^2$. We again get the phenomenon of frequency gaps corresponding to the right-hand side being less than zero. The width of the gap in the range $q_{\shortparallel}^2 V_0^2 > \omega^2 > q_{\shortparallel}^2 V_1^2$ is found to be

$$\Delta\omega = q_{\shortparallel} V_1 - q_{\shortparallel}\left[\frac{V_0 V_1 (2V_0 + V_1)}{V_0 + 2V_1}\right]^{1/2}. \quad (4.17)$$

(*ii*) *Strong fields.* When all the electrons are in the lowest Landau level, only the first term with $s=0$ remains in (4.10) in the zero-temperature limit. The dispersion relation is

$$1 = \frac{2\omega_p^2 m}{\hbar q^2} e^{-\frac{1}{2}a^2 q_\perp^2} \sum_{n=0}^{\infty} \frac{1}{n!}(\tfrac{1}{2}a^2 q_\perp^2)^n \times \frac{n\omega_c + \hbar q_{\shortparallel}^2/2m}{\omega^2 - (n\omega_c + \hbar q_{\shortparallel}^2/2m)^2}. \quad (4.18)$$

In the long-wavelength limit this reduces to (4.8). This dispersion relation is very similar to (4.13) and we will not discuss it further.

Spectrum of Fluctuations

The fluctuation spectrum can be obtained immediately from (2.12) and (4.10) with the result

$$F(\mathbf{q},\omega) = \frac{e^2 m}{2\pi\hbar q a^2} \coth(\tfrac{1}{2}\hbar\omega\beta) |1 - H(\mathbf{q},\omega)|^{-2} e^{-\frac{1}{2}a^2 q_\perp^2} \times \sum_{s=0}^{\infty} \sum_{n=0}^{\infty}{}' \frac{s!}{(n+s)!} (\tfrac{1}{2}a^2 q_\perp^2)^n [L_s^n(\tfrac{1}{2}a^2 q_\perp^2)]^2 \times [f_0(\epsilon_n^- + \hbar\omega_c(s+\tfrac{1}{2})) - f_0(\epsilon_{-n}^+ + \hbar\omega_c(s+\tfrac{1}{2})) + f_0(\epsilon_{-n}^- + \hbar\omega_c(s+n+\tfrac{1}{2})) - f_0(\epsilon_n^+ + \hbar\omega_c(s+n+\tfrac{1}{2}))], \quad (4.19)$$

where $H(\mathbf{q},\omega)$ is the right-hand side of (4.10) with ω replaced by $\omega + i\eta$. In (4.19) the prime on the summation means that for $n=0$ we must include only the first two terms in the square bracket and

$$\epsilon_n^{\pm} = (m/2q_{\shortparallel}^2)[\omega \pm (\hbar/2m) q_{\shortparallel}^2 - n\omega_c]^2.$$

In the limit of $q_{\shortparallel} \to 0$ from (4.19) the spectrum reduces to a set of discrete lines at $\omega = n\omega_c$ as in the Boltzmann case. At zero temperature f_0 is zero unless its argument is less than μ and this gives a restriction on the values of the frequency ω for which the various terms of (4.19) are nonzero and thus for which fluctuations exist. We illustrate these relations for the two cases of long wavelengths and strong magnetic fields.

(*i*) *Long wavelengths.* We only retain the terms with $n=0$ in (4.19) and in the zero-temperature limit we find

$$F(\mathbf{q},\omega) = \frac{e^2 m}{2\pi\hbar a^2 q} \coth(\tfrac{1}{2}\hbar\omega\beta) |1 - H(\mathbf{q},\omega)|^{-2} \sum_{s=0} C_s, \quad (4.20)$$

where the summation over s extends only over those integers for which $\mu > \hbar\omega_c s$. The function C_s is defined by (using $y = \hbar\omega/\mu$, $y_c = \hbar\omega_c/\mu$ and $x^2 = \hbar^2 q^2/2m\mu$),

$$C_0 = 1, \quad C_s = 2 \quad \text{if} \quad |x^2 - 2x(1 - s y_c)^{1/2}| < y < x^2 + 2x(1 - s y_c)^{1/2}. \quad (4.21)$$

To obtain the zero-field limit the summation in (4.20) is replaced by an integration and then using (4.21)

$$F(q,\omega) = \frac{e^2 m^2}{\pi\hbar^3 q} \coth(\tfrac{1}{2}\hbar\omega\beta) |1 - G(q,\omega)|^{-2} C, \quad (4.22)$$

where $G(q,\omega)$ is the right-hand side of (4.12) with $\omega \to \omega + i\eta$ and C is the following function:

$$\begin{aligned} &x<2, \quad 0 < y < 2x - x^2, \quad C = \mu y, \\ &|x^2 - 2x| < y < x^2 + 2x, \\ &C = (\mu/4x^2)(4x^2 - y^2 - x^4 + 2x^2 y). \end{aligned} \quad (4.23)$$

This result was first obtained by Hubbard.[16]

(*ii*) *Strong fields.* We only retain the first two terms with $s=0$ in (4.19). Taking the zero-temperature limit, we find

$$F(\mathbf{q},\omega) = \frac{e^2 m}{2\pi\hbar a^2 q} \coth(\tfrac{1}{2}\hbar\omega\beta) |1 - H(\mathbf{q},\omega)|^{-2} e^{-\frac{1}{2}a^2 q_\perp^2} \times \sum_{n=0}^{\infty} \frac{1}{n!} (\tfrac{1}{2}a^2 q_\perp^2)^n C_n', \quad (4.24)$$

where

$$C_n' = 1 \quad \text{if} \quad |y - n y_c - x^2| < 2x. \quad (4.25)$$

In this case there are fluctuations in the gas for all frequencies $\omega \sim n\omega_c$, i.e., multiples of the electron cyclotron frequency.

Finally, we obtain the damping of the plasma oscillations from the above results using (3.21) and (4.20) and (4.24). For long wavelengths,

$$\sigma(\omega) = -\frac{e^2 m \omega^3}{2\hbar^2 \omega_p^2 a^2 q^3} \sum_{s=0} C_s. \quad (4.26)$$

For strong fields,

$$\sigma(\omega) = -\frac{e^2 m \omega^3}{2\hbar^2 \omega_p^2 a^2 q^3} e^{-\frac{1}{2}a^2 q_\perp^2} \sum_{n=0}^{\infty} \frac{1}{n!} (\tfrac{1}{2}a^2 q_\perp^2)^n C_n'. \quad (4.27)$$

[16] J. Hubbard, Proc. Roy. Soc. (London) A 243, 336 (1957).